COMMON FIXED POINT RESULTS FOR MAPPINGS WITH RATIONAL EXPRESSIONS IN *B*-RECTANGULAR METRIC SPACES

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Abstract

In this article, we prove some common fixed point results for mappings involving certain rational expressions in complete b-rectangular metric spaces. In the process, we generalize various results of the literature.

Keywords*: b-rectangular metric spaces, common fixed point, weakly compatible mappings, complete b-rectangular metric spaces.*

1. Introduction

Fixed point theory is one of the traditional theory in mathematics and has a large number of applications in it and many branches of nonlinear analysis. Wide application potential of this theory has accelerated the research activities which resulted in enormous increase in publications. In a large class of studies the classical concept of metric space has been generalized in different directions by partly changing the conditions of the metric. Among these generalizations, one can mention the rectangular metric spaces defined by Branciari (Branciari, A. 2000)

In 2000, Branciari introduced the notion of generalized (rectangular) metric spaces where the triangle inequality of metric spaces was replaced by rectangular inequality. Later George and Radenovic (George, R., Radenovic, S., Reshma, K. P., Shukla, S. 2015) introduced the notion of b-rectangular metric spaces.

Several authors proved various fixed point results in such spaces generalizing the Banach contraction, Kannan type mappings etc. [see, e.g. (Ding, H.Sh., Ozturk, V., Radenovic, S. 2015)]

Here we prove a common fixed point result for self-mappings involving certain rational expressions in complete b-rectangular metric spaces.

2. Preliminaries

For the convenience, we start with the following definitions, lemmas and theorems.

Definition 1.1. (Bakhtin, I.A. 1989), Czerwik, S. (1993)] Let X be a (nonempty) set and $s\geq 1$ be a given real number. A function $d: X\times X \to [0,\infty[$ is a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:

(b1) $d(x,y) = 0$ if and only if $x = y$,
(b2) $d(x,y) = d(y,x)$
(b3) $d(x,y) \leq s[d(x,z) + d(z,y)]$

In this case, the pair (X,d) is called a b-metric spaces.

Definition 1.2. (Branciari, A. 2000) Let X be a (nonempty) set and let $d: X\times X \to [0,\infty[$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y:

(r1) $d(x,y) = 0$ if and only if $x = y$,
(r2) $d(x,y) = d(y,x)$,
(r3) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$

Then pair (X,d) is called a rectangular metric spaces (RMS) or Branciari'spaces [1].

Definition 1.3. (George, R., Radenovic, S., Reshma, K. P., Shukla, S. 2015) Let X be a (nonempty) set, $s \geq 1$ be a given real number and let $d: X \times X \to [0, \infty[$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y:
(br1) $d(x, y) = 0$ if and only if $x = y$,
(br2) $d(x, y) = d(y, x)$,
(br3) $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$

Then pair (X, d) is called a b-rectangular metric spaces (b-RMS)

Note that every metric spaces is a rectangular metric space and every rectangular metric space is a b-rectangular metric space (with $s = 1$). However the converse is not necessarily true. The following example, is inspired by (Shukla, S. 2014)., shows that $(X, d)$is a RMS, which is not metric spaces.

Example 1.4. Let $X = \{\frac{1}{n} : n \in N\} \cup \{0\}$. Define $d: X \times X \to [0, \infty[$ as follows:

$$d(x, y) = \begin{cases} 0 & x = y \\ \frac{1}{n} & \{x, y\} = \{0, \frac{1}{n}\} \\ 1 & x \neq y,\ x,\ y \in X / \{0\} \end{cases}$$

Then it is easy to check that d is rectangular metric and is not a metric since

$$d(\frac{1}{6}, \frac{1}{10}) = 1 > \frac{1}{6} + \frac{1}{10} = d(\frac{1}{6}, 0) + d(0, \frac{1}{10})$$

Also, the following example shows that a b-rectangular metric space is not a rectangular metric space.

Example 1.5 Let $X = N$. Define $d: X \times X \to [0, \infty[$ as follows

$$d(x, y) = \begin{cases} 0 & x = y \\ 4\alpha & x, y \in \{1, 2\} \text{ and } x \neq y \\ \alpha & x \text{ or } y \notin \{1, 2\} \text{ and } x \neq y \end{cases}$$

where $\alpha > 0$ is a constant. Than (X, d) is a b-rectangular metric space with s$==\frac{4}{3} > 1$ but it is not a rectangular metric space, as $d(1,2) = 4\alpha > 3\alpha = d(1,3) + d(3,4) + d(4,2)$.

Note that every b-metric space with coefficient s is a b-rectangular metric space with coefficient s^2, but the converse is not necessarily true. See the following example.

Example 1.6. (George, R., Radenovic, S., Reshma, K. P., Shukla, S. 2015). Let $X = A \cup B$, where A= $\{\frac{1}{n} : n \in N\}$ and B is the set of positive integers. Define $d: X \times X \to [0, \infty[$ such that $d(x, y) = d(y, x)$ and

$$d(x, y) = \begin{cases} 0 & x = y \\ 2\alpha & x, y \in A \\ \frac{\alpha}{2n} & x \in A \text{ and } y \in \{2, 3\} \\ \alpha & \text{atherwise} \end{cases}$$

where $\alpha > 0$ is a constant. Than (X, d) is a b-rectangular metric space with s=2>1, but it is not a b- metric space, as for every $x = \frac{1}{n}, y = \frac{1}{m}$ does not exist $s > 0$ such that

$$d(x, y) = d(\frac{1}{n}, \frac{1}{m}) = 2\alpha \leq s[d(x, 2) + d(2, y)] = s\left[\frac{\alpha}{2n} + \frac{\alpha}{2m}\right]$$

Definition 1.7. Let *(X, d)* be a rectangular b-metric space, $\{xn\}$ be a sequence in X and $x \in X$. Then

(a) The sequence $\{x_n\}_{n\in N}$ is called a b-RMS Cauchy sequence if and only if for each ε>0, there exists a natural number $n_0 \in N$ such that for $m>n>n_0$, we have $d(x_m, x_n)<\varepsilon$ or equivalently, if $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$.

(b) The sequence $\{x_n\}_{n\in N}$ is said to be b-RMS convergent in $x\in X$ *if* and only if for every ε>0 there exists $n_0 \in N$ such that for $n>n_0$, we have $d(x_n, x)<\varepsilon$ or equivalently, if $\lim\limits_{n\to\infty} d(x_n, x) = 0$.

(c) A b -rectangular metric space (X,d) is called complete if every Cauchy sequence in it is b- RMS converges to some $x \in X$.

Note that, limit of a sequence in a b-RMS is not necessarily unique and also every b-RMS-convergent sequence in a b-RMS is not necessarily b-RMS-Cauchy. The example 1.6 show that fact. (The sequence $\{\frac{1}{n}\}$ converges in 2 and in 3, so the limit is not unique. Also it is not Cauchy.).

Lemma 1.8. (Ding, H.Sh., Ozturk, V., Radenovic, S. 2015) Let *(X, d)* be a rectangular b-metric space with $s \geq 1$ and let $\{x_n\}$ be a sequence in X such that $x_n \neq x_m$ whenever $n \neq m$ and $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$.

If $\{x_n\}$ is not a b-rectangular Cauchy sequences, the there exist ε>0 and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that for the following sequences of real numbers $d(x_{m(k)}, x_{n(k)})$, $d(x_{m(k)+1}, x_{n(k)-1})$ and $d(x_{m(k)}, x_{n(k)-2})$ hold:

$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$ $\quad \frac{\varepsilon}{s} \leq \overline{\lim\limits_{k\to\infty}} d(x_{m(k)}, x_{n(k)-2}) \leq \varepsilon$ and $\frac{\varepsilon}{s} \leq \overline{\lim\limits_{k\to\infty}} d(x_{m(k)+1}, x_{n(k)-1})$

Lemma 1.9. (Ding, H.Sh., Ozturk, V., Radenovic, S. 2015) Let *(X, d)* be a rectangular b-metric space with $s \geq 1$ and let $\{x_n\}$ be a b-rectangular-Cauchy sequence in *X* such that $x_n \neq x_m$ whenever $n \neq m$. Then $\{x_n\}$ can converge at most one point.

3. Main Results

Theorem 2.1. Let *(X, d)* be a rectangular b-metric space with $s \geq 1$, and $f, g: X \to X$ be two self maps such that for all $x, y \in X$

(1)
$$d(fx, gy) \leq \alpha d(gx, gy) + \beta \frac{d(gx, fx)d(gy, fy)}{d(gx, gy)} + \gamma\left[d(gx, fx) + d(gy, fy)\right] + \delta\left[d(gx, fx) + d(gx, gy)\right] + \eta\left[d(gy, fy) + d(gx, gy)\right] + kd(gx, fy)$$

where $0 < \alpha + \beta + 2\gamma + 2\delta + 2\eta + k < 1$, $0 \leq \alpha + \delta + \eta + k < \frac{1}{s}$ and $0 \leq \gamma + \delta < \frac{1}{s}$.

(2) $f(X) \subseteq g(X)$, one of these two subset of X being complete

(3) f, g are weakly compatible

then f, g have a unique common fixed point.

Proof. Let x_0 be a arbitrary point of X. Since $f(X) \subseteq g(X)$ we can choose $x_1 \in X$ such that $f(x_0) = g(x_1)$. Countinuing this process, having choosen x_n in X, we obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. So, for convenience, let put $y_n = fx_n = gx_{n+1}$.
If for any n, $y_n = y_{n+1}$, then $fx_{n+1} = gx_{n+1}$ and f, g have a point of coincidence.
Suppose that $y_n \neq y_{n+1}$ for all $n \in N$. From (1) in theorem we obtain that

$$\begin{aligned}
d(y_{n+1}, y_n) &= d(fx_{n+1}, fx_n) \le \alpha d(gx_{n+1}, gx_n) + \beta \frac{d(gx_{n+1}, fx_{n+1}) d(gx_n, fx_n)}{d(gx_{n+1}, gx_n)} \\
&\quad + \gamma\left[d(gx_{n+1}, fx_n) + d(gx_n, fx_{n+1})\right] + \delta\left[d(gx_{n+1}, fx_{n+1}) + d(gx_{n+1}, gx_n)\right] \\
&\quad + \eta\left[d(gx_n, fx_n) + d(gx_{n+1}, gx_n)\right] + kd(gx_{n+1}, fx_{n+1}) \\
&= \alpha d(y_n, y_{n-1}) + \beta \frac{d(y_n, y_{n+1}) d(y_{n-1}, y_n)}{d(y_n, y_{n-1})} + \gamma\left[d(y_n, y_{n+1}) + d(y_{n-1}, y_n)\right] \\
&\quad + \delta\left[d(y_n, y_{n+1}) + d(y_n, y_{n-1})\right] + \eta\left[d(y_{n-1}, y_n) + d(y_n, y_{n-1})\right] + kd(y_n, y_{n+1}) \\
&= \alpha d(y_n, y_{n-1}) + \beta d(y_n, y_{n+1}) + \gamma d(y_n, y_{n+1}) + \gamma d(y_{n-1}, y_n) + \delta d(y_{n+1}, y_n) + \delta d(y_n, y_{n-1}) \\
&\quad + 2\eta d(y_n, y_{n-1}) + kd(y_{n+1}, y_n)
\end{aligned}$$

that is $d(y_{n+1}, y_n) \le \lambda d(y_n, y_{n-1})$ where $\lambda = \dfrac{\alpha + \gamma + \delta + 2\eta}{1 - \beta - \gamma - \delta - k}$ and $0 \le \lambda < 1$

So, $d(y_{n+1}, y_n) \le \lambda d(y_n, y_{n-1}) \le \lambda^2 d(y_{n-1}, y_{n-2}) \le \ldots \le \lambda^n d(y_0, y_1)$.
Since $0 \le \lambda < 1$, then we have that

$$\lim_{n\to\infty} d(y_{n+1}, y_n) = 0 \tag{4}$$

Now, let us prove that sequence $\{y_n\}$ is a b-rectangular- Cauchy sequence in b-RMS (X,d) . The sequence $\{y_n\}$ satisfies all conditions of Lemma 1.8. So, if we suppose that it is not a b-rectangular-Cauchy sequence, then there exist $\varepsilon > 0$ and two subsequences $\{y_{m(k)}\}$ and $\{y_{n(k)}\}$ such that the following sequences of real numbers $d\left(y_{m(k)}, y_{n(k)}\right)$, $d\left(y_{m(k)+1}, y_{n(k)-1}\right)$ and $d\left(y_{m(k)}, y_{n(k)-2}\right)$ satisfy the relations of Lemma 8. The index $n(k)$ is the smallest index for which $d\left(y_{m(k)}, y_{n(k)}\right) \ge \varepsilon$. So, by (1) we have

$$\begin{aligned}
d(y_{m(k)+1}, y_{n(k)-1}) &= d(fx_{m(k)+1}, fx_{n(k)-1}) \le \alpha d(gx_{m(k)+1}, gx_{n(k)-1}) \\
&\quad + \beta \frac{d(gx_{m(k)+1}, fx_{m(k)+1}) d(gx_{n(k)-1}, fx_{n(k)-1})}{d(gx_{m(k)+1}, gx_{n(k)-1})} + \gamma\left[d(gx_{m(k)+1}, fx_{m(k)+1}) + d(gx_{n(k)-1}, fx_{n(k)-1})\right] \\
&\quad + \delta\left[d(gx_{m(k)+1}, fx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{n(k)-1})\right] \\
&\quad + \eta\left[d(gx_{n(k)-1}, fx_{n(k)-1}) + d(gx_{m(k)+1}, gx_{n(k)-1})\right] + kd(gx_{m(k)+1}, fx_{n(k)-1}) \\
&= \alpha d(y_{m(k)}, y_{n(k)-2}) + \beta \frac{d(y_{m(k)}, y_{m(k)+1}) d(y_{n(k)-2}, y_{n(k)-1})}{d(y_{m(k)}, y_{n(k)-2})} + \\
&\quad \gamma\left[d(y_{m(k)}, y_{m(k)+1}) + d(y_{n(k)-2}, y_{n(k)-1})\right] + \delta\left[d(y_{m(k)}, y_{m(k)+1}) + d(y_{m(k)}, y_{n(k)-2})\right] \\
&\quad + \eta\left[d(y_{n(k)-2}, y_{n(k)-1}) + d(y_{m(k)}, y_{n(k)-2})\right] + kd(y_{m(k)}, y_{n(k)-1})
\end{aligned}$$

Taking the upper limit as $k \to \infty$ and using Lemma 8 and (4) we get

$$\frac{\varepsilon}{s} \le \alpha\varepsilon + \beta \cdot 0 + \gamma \cdot 0 + \delta\varepsilon + \eta\varepsilon + k\varepsilon = \left(\alpha + \delta + \eta + k\right)\varepsilon$$

that is a contradiction, because $\alpha + \delta + \eta + k \le \frac{1}{s}$. Hence the sequence $\{y_n\}$ is a b-rectangular-Cauchy sequence in b-RMS (X,d).

Suppose, without loss generality, that $g(X)$ is complete. Then $\{y_n\}$ converges to some $u \in g(X)$, so exists $v \in X$ such that $u = gv$.

To prove that $fv = gv$. Suppose that $fv \neq gv$ and by Lemma 1.9, $y_n = fx_n \neq fv$ and $y_n = gx_{n+1} \neq gv$ for n sufficiently large. Hence, by b-rectangular inequality we have

$$\begin{aligned} d(fv,gv) &\le s[d(fv,fx_n) + d(fx_n,fx_{n+1}) + d(fx_{n+1},gv)] \\ &= s[d(fv,fx_n) + d(y_n,y_{n+1}) + d(y_{n+1},gv)] \end{aligned} \tag{5}$$

$$\begin{aligned} d(fv,fx_n) &\le \alpha d(gv,gx_n) + \beta \frac{d(gv,fv)d(gx_n,fx_n)}{d(gv,gx_n)} \\ &\quad + \gamma\left[d(gv,fv) + d(gx_n,fx_n)\right] + \delta\left[d(gv,fv) + d(gv,gx_n)\right] \\ &\quad + \eta\left[d(gx_n,fx_n) + d(gv,gx_n)\right] + kd(gv,fx_n) \\ &= \alpha d(u,y_{n-1}) + \beta \frac{d(u,fv)d(y_{n-1},y_n)}{d(u,y_{n-1})} + \gamma\left[d(u,fv) + d(y_{n-1},y_n)\right] \\ &\quad + \delta\left[d(u,fv) + d(u,y_{n-1})\right] + \eta\left[d(y_{n-1},y_n) + d(u,y_{n-1})\right] + kd(u,y_n) \end{aligned} \tag{6}$$

By (5) and (6) we have

$$\begin{aligned} \frac{1}{s}d(fv,u) &\le \alpha d(u,y_{n-1}) + \beta \frac{d(u,fv)d(y_{n-1},y_n)}{d(gv,y_{n-1})} + \gamma\left[d(u,fv) + d(y_{n-1},y_n)\right] \\ &\quad + \delta\left[d(u,fv) + d(u,y_{n-1})\right] + \eta\left[d(y_{n-1},y_n) + d(u,y_{n-1})\right] + kd(u,y_n) + d(y_n,y_{n+1}) + d(y_{n+1},u) \end{aligned} \tag{7}$$

Taking the limit as $n \to \infty$, we get

$$\frac{1}{s}d(fv,u) \le \alpha \cdot 0 + \beta \cdot 0 + \gamma d(u,fv) + \delta d(u,fv) + \eta \cdot 0 + k \cdot 0 + 0 = (\gamma + \delta)d(fv,u)$$

Since $\gamma + \delta < \frac{1}{s}$ we have $d(fv,u) = 0$ and $fv = u = gv$.

But the maps f,g are weakly compatible, thus $fu = fgv = gfv = gu$.

Let us show that u is a fixed point for f and g.

$$\begin{aligned} d(fu,y_n) = d(fu,fx_n) &\le \alpha d(gu,gx_n) + \beta \frac{d(gu,fu)d(gx_n,fx_n)}{d(gu,gx_n)} \\ &\quad + \gamma\left[d(gu,fu) + d(gx_n,fx_n)\right] + \delta\left[d(gu,fu) + d(gu,gx_n)\right] \\ &\quad + \eta\left[d(gx_n,fx_n) + d(gu,gx_n)\right] + kd(gu,fx_n) \\ &= \alpha d(fu,y_{n-1}) + \beta \frac{d(fu,fu)d(y_{n-1},y_n)}{d(fu,y_{n-1})} + \gamma\left[d(fu,fu) + d(y_{n-1},y_n)\right] \\ &\quad + \delta\left[d(fu,fu) + d(fu,y_{n-1})\right] + \eta\left[d(y_{n-1},y_n) + d(fu,y_{n-1})\right] + kd(fu,y_n) \end{aligned} \tag{8}$$

Taking the limit as $n \to \infty$, we get

$$d(fu,u) \le \alpha d(fu,u) + \delta d(fu,u) + \eta d(fu,u) + kd(fu,u) = (\alpha + \delta + \eta + k)d(fu,u) \tag{9}$$

Since $\alpha + \delta + \eta + k < 1$ we get $d(fu,u) = 0$ and $fu = u = gu$ i.e. u is a common fixed point for f and g.

We can prove that u is unique. In fact, if u and v are two common fixed points for f and g and by (1) we have

$$d(u,v) = d(fu, fv) \le \alpha d(gu, gv) + \beta \frac{d(gu, fu)d(gv, fv)}{d(gu, gv)} + \gamma\left[d(gu, fu) + d(gv, fv)\right]$$

$$+\delta\left[d(gu, fu) + d(gu, gv)\right] + \eta\left[d(gv, fv) + d(gu, gv)\right] + kd(gu, fv)$$

$$= \alpha d(u,v) + \delta d(u,v) + \eta d(u,v) + kd(u,v) = (\alpha+\delta+\eta+k)d(u,v)$$

Since $\alpha + \delta + \eta + k < 1$ we get $d(u, v) = 0$ and u=v. The proof is complete.

Remark 2.2. For $\alpha = a, \gamma = b$ and $\beta = \delta = \eta = k = 0$ in Theorem 2.1 we have the theorem of Ding et al. in [2].

If $g = I_X$ we take the following theorem:

Theorem 2.3. Let *(X, d)* be a rectangular b-metric space with $s \geq 1$, and $f: X \to X$ be a self map such that for all $x, y \in X$

(1)
$$d(fx, fy) \le \alpha d(x,y) + \beta \frac{d(x, fx)d(y, fy)}{d(x,y)} + \gamma\left[d(x, fx) + d(y, fy)\right]$$

$$+\delta\left[d(x, fx) + d(x,y)\right] + \eta\left[d(y, fy) + d(x,y)\right] + kd(x, fy)$$

where $0 < \alpha + \beta + 2\gamma + 2\delta + 2\eta + k < 1$, $0 \leq \alpha + \delta + \eta + k < \frac{1}{s}$ and $0 \leq \gamma + \delta < \frac{1}{s}$.

(2) $f(X)$ is complete, then f have a unique fixed point.

This theorem generalizes the corollaries 2.2, 2.3, 2.4, 2.5 in (Ding, H.Sh., Ozturk, V., Radenovic, S. 2015).

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